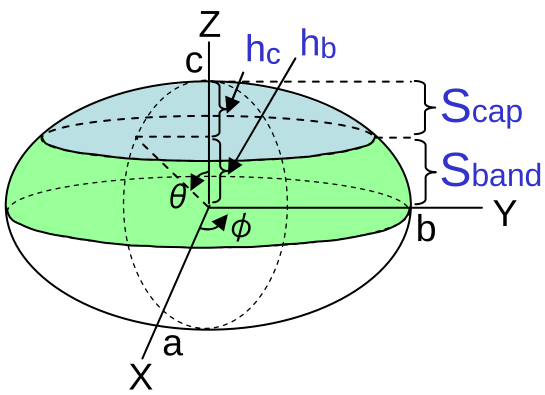
How to calculates the surface area of an ellipsoidal cap by numerical integration

2015. 10.29 keisan



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

surface area of an ellipsoidal band S_{band} $0 \leqq z \leqq h_b \leqq c$

surface area of an ellipsoidal cap S_{cap} $0 \leqq c-h_c \leqq z \leqq c$

Gauss-Chebyshev integration

$$I=\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}dx=\sum_{i=1}^{n}w_i f(x_i)$$

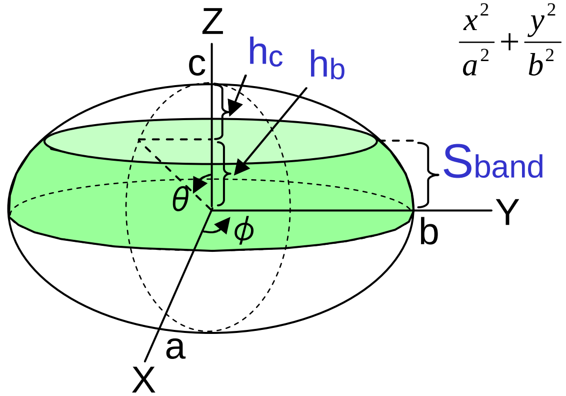
$$w_i=\frac{\pi}{n},\quad x_i=\cos\left(\frac{2i-1}{2n}\pi\right)$$

1) No restriction on the size of a, b and c by a complex calculation

2) Ensure the accuracy of S_{cap} and S_{band} even in the case of hc<<c or hb<<c
（only calculation of keisan ）

3) Use the Gauss-Chebyshev integration（Easy calculation of Nodes and Weights ）

surface area of an ellipsoidal band



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$x=a\sin\theta\cos\varphi$$

$$y=b\sin\theta\sin\varphi$$

$$z=c\cos\theta$$

$$\mathbf{r}=a\sin\theta\cos\varphi\,\mathbf{i}+b\sin\theta\sin\varphi\,\mathbf{j}+c\cos\theta\,\mathbf{k}$$

$$d\mathbf{S}=\frac{\partial\mathbf{r}}{\partial\theta}d\theta\times\frac{\partial\mathbf{r}}{\partial\varphi}d\varphi=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ a\cos\theta\cos\varphi & b\cos\theta\sin\varphi & -c\sin\theta\\ -a\sin\theta\sin\varphi & b\sin\theta\cos\varphi & 0\end{vmatrix}d\theta d\varphi$$

$$=\left(bc\sin\theta\cos\varphi\,\mathbf{i}+ca\sin\theta\sin\varphi\,\mathbf{j}+ab\cos\theta\,\mathbf{k}\right)\sin\theta\,d\theta d\varphi$$

$$dS=\left|d\mathbf{S}\right|=\sqrt{b^2c^2\sin^2\theta\cos^2\varphi+c^2a^2\sin^2\theta\sin^2\varphi+a^2b^2\cos^2\theta}\,\sin\theta\,d\theta d\varphi$$

$$S_{band}=\iint dS$$

$$=\int_{\theta=\cos^{-1}h}^{\frac{\pi}{2}}\int_{\varphi=0}^{2\pi}\sqrt{b^2c^2\sin^2\theta\cos^2\varphi+c^2a^2\sin^2\theta\sin^2\varphi+a^2b^2\cos^2\theta}\,\sin\theta d\theta d\varphi$$

$$=4bc\int_{\varphi=0}^{\frac{\pi}{2}}\sqrt{B}\int_{u=0}^{h}\sqrt{1+Fu^2}\,dud\varphi$$

$$=2bc\int_0^{\frac{\pi}{2}}\sqrt{B}\left|u\sqrt{1+Fu^2}+\frac{1}{\sqrt{F}}\sinh^{-1}\left(\sqrt{F}u\right)\right|_0^h d\varphi$$

$$=2bch\int_0^{\frac{\pi}{2}}\sqrt{B}\left(\sqrt{1+Fh^2}+\frac{\sinh^{-1}\left(\sqrt{F}h\right)}{\sqrt{F}h}\right)d\varphi$$

$$=bch\int_{-1}^{1}\sqrt{B(t)}\left(\sqrt{1+F(t)h^2}+\frac{\sinh^{-1}\left(h\sqrt{F(t)}\right)}{h\sqrt{F(t)}}\right)\frac{dt}{\sqrt{1-t^2}}$$

$$=bch\sum_{i=1}^{n}\frac{\pi}{n}f\left(\cos\left(\frac{2i-1}{2n}\pi\right)\right)$$

$$u=\cos\theta,\quad du=-\sin\theta d\theta,\quad h=\frac{h_b}{c}$$

$$B=\cos^2\varphi+\frac{a^2}{b^2}\sin^2\varphi=1-\left(1-\frac{a^2}{b^2}\right)\sin^2\varphi,\quad F=\frac{a^2}{c^2}\frac{1}{B}-1$$

$$\int\sqrt{1+ax^2}\,dx=\frac{x}{2}\sqrt{1+ax^2}+\frac{1}{2\sqrt{a}}\sinh^{-1}\left(\sqrt{a}\,x\right)$$

$$t=\sin\varphi,\quad d\varphi=\frac{dt}{\sqrt{1-t^2}}$$

$$B(t)=1-\left(1-\frac{a^2}{b^2}\right)t^2,\quad F(t)=\frac{a^2}{c^2}\frac{1}{B(t)}-1$$

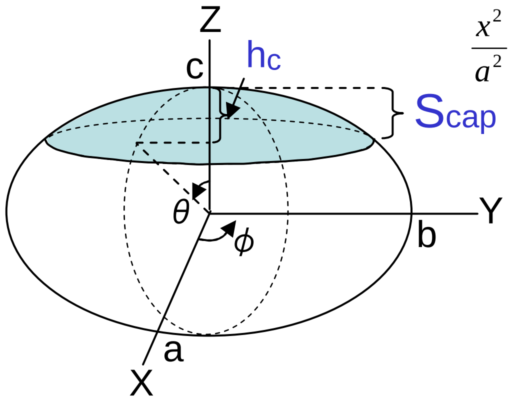
$$f(t)=\sqrt{B(t)}\left(\sqrt{1+F(t)h^2}+\frac{\sinh^{-1}\left(h\sqrt{F(t)}\right)}{h\sqrt{F(t)}}\right)$$

$$F(t)=0\ \Rightarrow\quad f(t)=2h\frac{a}{c}$$

Gauss-Chebyshev integration

$$S=bch\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}dx=bch\sum_{i=1}^{n}w_i f(x_i)\qquad w_i=\frac{\pi}{n},\quad x_i=\cos\left(\frac{2i-1}{2n}\pi\right)$$

surface area of an ellipsoidal cap



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$x=a\sin\theta\cos\varphi$$
$$y=b\sin\theta\sin\varphi$$
$$z=c\cos\theta$$

$$\mathbf{r}=a\sin\theta\cos\varphi\,\mathbf{i}+b\sin\theta\sin\varphi\,\mathbf{j}+c\cos\theta\,\mathbf{k}$$

$$d\mathbf{S}=\frac{\partial\mathbf{r}}{\partial\theta}d\theta\times\frac{\partial\mathbf{r}}{\partial\varphi}d\varphi=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a\cos\theta\cos\varphi & b\cos\theta\sin\varphi & -c\sin\theta\\ -a\sin\theta\sin\varphi & b\sin\theta\cos\varphi & 0\end{vmatrix}d\theta d\varphi$$

$$=\left(bc\sin\theta\cos\varphi\,\mathbf{i}+ca\sin\theta\sin\varphi\,\mathbf{j}+ab\cos\theta\,\mathbf{k}\right)\sin\theta\,d\theta d\varphi$$

$$dS=\left|d\mathbf{S}\right|=\sqrt{b^2c^2\sin^2\theta\cos^2\varphi+c^2a^2\sin^2\theta\sin^2\varphi+a^2b^2\cos^2\theta}\,\sin\theta\,d\theta d\varphi$$

$$S_{cap}=\iint dS$$

$$=\int_{\theta=0}^{\cos^{-1}h}\int_{\varphi=0}^{2\pi}\sqrt{b^2c^2\sin^2\theta\cos^2\varphi+c^2a^2\sin^2\theta\sin^2\varphi+a^2b^2\cos^2\theta}\,\sin\theta d\theta d\varphi$$

$$=4bc\int_{\varphi=0}^{\frac{\pi}{2}}\sqrt{B}\int_{u=h}^{1}\sqrt{1+Fu^2}\,du d\varphi$$

$$=2bc\int_0^{\frac{\pi}{2}}\sqrt{B}\left|u\sqrt{1+Fu^2}+\frac{1}{\sqrt{F}}\sinh^{-1}\left(\sqrt{F}u\right)\right|_h^1 d\varphi$$

$$=2bc\int_0^{\frac{\pi}{2}}\sqrt{B}\left\{\sqrt{1+F}-h\sqrt{1+Fh^2}+\frac{1}{\sqrt{F}}\left(\sinh^{-1}\left(\sqrt{F}\right)-\sinh^{-1}\left(h\sqrt{F}\right)\right)\right\}d\varphi$$

$$=2bc\int_0^{\frac{\pi}{2}}\sqrt{B}\left\{\sqrt{1+F}-h\sqrt{1+Fh^2}+\frac{1}{\sqrt{F}}\sinh^{-1}\left(\sqrt{F}\left(\sqrt{1+Fh^2}-h\sqrt{1+F}\right)\right)\right\}d\varphi$$

$$=ab\int_{-1}^{1}\left\{1-hG(t)+\frac{c}{a}\sqrt{\frac{B}{F}}\sinh^{-1}\left(\frac{a}{c}\sqrt{\frac{F}{B}}\left(G(t)-h\right)\right)\right\}\frac{dt}{\sqrt{1-t^2}}$$

$$=ab\int_{-1}^{1}\left\{1-hG(t)+\left(G(t)-h\right)\frac{\sinh^{-1}\left(\frac{a}{c}\sqrt{\frac{F}{B}}\left(G(t)-h\right)\right)}{\frac{a}{c}\sqrt{\frac{F}{B}}\left(G(t)-h\right)}\right\}\frac{dt}{\sqrt{1-t^2}}$$

$$=ab\sum_{i=1}^{n}\frac{\pi}{n}f\left(\cos\left(\frac{2i-1}{2n}\pi\right)\right)$$

$$u=\cos\theta,\quad du=-\sin\theta d\theta\qquad h=1-\frac{h_c}{c}$$

$$B=\cos^2\varphi+\frac{a^2}{b^2}\sin^2\varphi=1-\left(1-\frac{a^2}{b^2}\right)\sin^2\varphi,\quad F=\frac{a^2}{c^2}\frac{1}{B}-1$$

$$\int\sqrt{1+ax^2}\,dx=\frac{x}{2}\sqrt{1+ax^2}+\frac{1}{2\sqrt{a}}\sinh^{-1}\left(\sqrt{a}\,x\right)$$

$$\sinh^{-1}x-\sinh^{-1}y=\sinh^{-1}\left(x\sqrt{1+y^2}-y\sqrt{1+x^2}\right)$$

$$t=\sin\varphi,\quad d\varphi=\frac{dt}{\sqrt{1-t^2}}\qquad B(t)=1-\left(1-\frac{a^2}{b^2}\right)t^2,\quad F(t)=\frac{a^2}{c^2}\frac{1}{B(t)}-1$$

$$G(t)=\sqrt{h^2+\left(1-h^2\right)\frac{c^2}{a^2}B(t)}$$

$$f(t)=1-hG(t)+\frac{c}{a}\sqrt{\frac{B}{F}}\sinh^{-1}\left(\frac{a}{c}\sqrt{\frac{F}{B}}\left(G(t)-h\right)\right)$$

$$F(t)=0\Rightarrow\quad f(t)=2(1-h)$$

Gauss-Chebyshev integration

$$S=ab\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}dx=ab\sum_{i=1}^{n}w_i f(x_i)\qquad w_i=\frac{\pi}{n},\quad x_i=\cos\left(\frac{2i-1}{2n}\pi\right)$$