How to calculates the volume of an ellipsoidal cap

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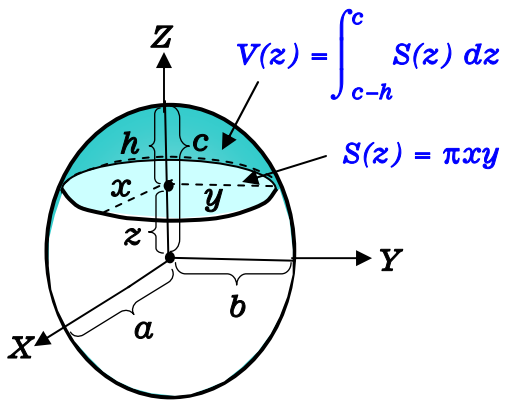

equation of an ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

equation of an ellipse along the X-axis (y=0)

$$\frac{x^2}{a^2}+\frac{z^2}{c^2}=1 \quad \Rightarrow \quad x=\frac{a}{c}\sqrt{c^2-z^2}$$

equation of an ellipse along the Y-axis (x=0)

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \Rightarrow \quad y=\frac{b}{c}\sqrt{c^2-z^2}$$

area of a vertical ellipse to the Z-axis

$$S(z)=\pi xy$$

$$=\pi ab\left(1-\frac{z^2}{c^2}\right)$$

volume of an ellipsoidal cap

$$V(z)=\int_{c-h}^{c} S(z)dz=\int_{c-h}^{c}\pi ab\left(1-\frac{z^2}{c^2}\right)dz$$

$$=\pi ab\left\{c-\frac{c^3}{3c^2}-(c-h)+\frac{(c-h)^3}{3c^2}\right\}$$

$$=\frac{\pi abh^2}{3c^2}(3c-h)$$

volume of an ellipsoid

$$V(z)=\int_{-c}^{c} S(z)dz=\int_{-c}^{c}\pi ab\left(1-\frac{z^2}{c^2}\right)dz$$

$$=\frac{4\pi abc}{3}$$